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Abstract

We prove that the injectively ω-tree-automatic ordinals are the ordinals smaller than ω^{ω^ω}. Then we show that the injectively ω^n-automatic ordinals, where $n \geq 1$ is an integer, are the ordinals smaller than ω^{ω^n}. This strengthens a recent result of Schlicht and Stephan who considered in [SS11] the subclasses of *finite word* ω^n-automatic ordinals. As a by-product we obtain that the hierarchy of injectively ω^n-automatic structures, $n \geq 1$, which was considered in [FT12], is strict.

Keywords: ω-tree-automatic structures; ω^n-automatic structures; ordinals.

1 Introduction

An automatic structure is a relational structure whose domain and relations are recognizable by finite automata reading finite words. Automatic structures have very nice decidability and definability properties and have been much studied in the last few years, see [BG00, BG04, KNRS07, Rub04, Rub08]. Blumensath considered in [Blu99] more powerful kinds of automata. If we replace automata by tree automata (respectively, Büchi automata reading infinite words, Muller or Rabin tree automata reading infinite labelled trees) then we get the notion of tree-automatic (respectively, ω-automatic, ω-tree-automatic) structures. Notice that an ω-automatic or ω-tree-automatic structure may have uncountable cardinality. All these kinds of automatic structures have the two following fundamental properties. (1) The class of automatic (respectively, tree-automatic, ω-automatic, ω-tree-automatic) structures is closed under first-order interpretations. (2) The first-order theory of an automatic (respectively, tree-automatic, ω-automatic, ω-tree-automatic) structure is decidable.

A natural problem is to classify firstly automatic structures using some invariants. For instance Delhommé proved that the automatic ordinals are the ordinals smaller than ω^ω, and that the tree-automatic ordinals are the ordinals smaller than ω^{ω^ω}, see [Del04, Rub04, Rub08]. Kuske proved in [Kus10] that the ω-automatic ordinals are the automatic ordinals, i.e. the ordinals smaller than ω^ω.

In the first part of this paper we characterize the (injectively) ω-tree automatic ordinals, proving that they are the ordinals smaller than ω^{ω^ω}. This seems to be the first complete characterization of a class of ω-tree automatic structures. The proof uses some results of Niwinski [Niw91] on the cardinality of regular tree languages, a recent result of Barany, Kaiser and Rabinovich [BKR09] on (injectively) ω-tree automatic structures, the result of Delhommé on tree automatic ordinals, and some set theory.

The ω^n-automatic structures, presentable by automata reading ordinals words of length ω^n, where $n \geq 1$ is an integer, have been recently investigated in [FT12]. We show here that the injectively ω^n-automatic ordinals are the ordinals smaller than ω^{ω^n}. This strengthens a recent result of Schlicht and Stephan who considered in [SS11] the subclasses of *finite word* ω^n-automatic ordinals. As a by-product we obtain that the hierarchy of injectively ω^n-automatic structures, $n \geq 1$, which was considered in [FT12], is strict.

The paper is organized as follows. In Section 2 we recall basic notions and some properties of automatic structures. We characterize the (injectively) ω-tree automatic ordinals in Section 3. We consider the ω^n-automatic ordinals in Section 4. Some concluding remarks are given in Section 5.

2 Automatic structures

When Σ is a finite alphabet, a *non-empty finite word* over Σ is any sequence $x = a_1 a_2 \ldots a_k$, where $a_i \in \Sigma$ for $i = 1, \ldots, k$, and k is an integer ≥ 1. The *length* of x is k. The *empty word* has no letter and is denoted by ε; its length is 0. For $x = a_1 a_2 \ldots a_k$, we write $x(i) = a_i$. $\Sigma^\star$ is the *set of finite words* (including the empty word) over Σ.

We introduce now languages of infinite binary trees whose nodes are labelled in a finite alphabet Σ. A node of an infinite binary tree is represented by a finite word over the alphabet $\{l, r\}$ where r means "right" and l means "left". Then an infinite binary tree whose nodes are labelled in Σ is identified with a function $t : \{l, r\}^\star \to \Sigma$. The set of infinite binary trees labelled in Σ will be denoted T_Σ^ω. A tree language is a subset of T_Σ^ω, for some alphabet Σ. (Notice that we shall mainly consider in the sequel *infinite* trees so we shall often use the term tree instead of *infinite* tree).

We assume the reader has some knowledge about Muller or Rabin tree automata and regular tree languages. We recall that the classes of tree languages accepted by non-deterministic Muller, Rabin, Street, and parity tree automata are all the same. We refer for instance to [Tho90, PP04, GTW02] for the definition of these acceptance conditions.

Notice that one can consider a relation $R \subseteq T_{\Sigma_1}^\omega \times T_{\Sigma_2}^\omega \times \ldots \times T_{\Sigma_k}^\omega$, where $\Sigma_1, \Sigma_2, \ldots \Sigma_k$, are finite alphabets, as a tree language over the product alphabet $\Sigma_1 \times \Sigma_2 \times \ldots \times \Sigma_k$.

Let now $\mathcal{M} = (M, (R_i^M)_{1 \leq i \leq k})$ be a relational structure, where M is the domain, and for each $i \in [1, k]$ R_i^M is a relation of finite arity n_i on the domain M. The structure is said to be ω-tree-automatic if there is a presentation of the structure where the domain and the relations on the domain are accepted by Muller tree automata, in the following sense.

Definition 2.1 (see [Blu99]) *Let $\mathcal{M} = (M, (R_i^M)_{1 \leq i \leq k})$ be a relational structure, where $n \geq 1$ is an integer, and each relation R_i is of finite arity n_i. An ω-tree-automatic presentation of the structure $\mathcal{M}$ is formed by a tuple of Muller tree automata $(\mathcal{A}, \mathcal{A}_=, (\mathcal{A}_i)_{1 \leq i \leq k})$, and a mapping h from $L(\mathcal{A})$ onto M, such that:*

1. *The automaton $\mathcal{A}_=$ accepts an equivalence relation $E_\equiv$ on $L(\mathcal{A})$, and*

2. *For each $i \in [1, k]$, the automaton $\mathcal{A}_i$ accepts an n_i-ary relation R'_i on $L(\mathcal{A})$ such that $E_\equiv$ is compatible with R'_i, and*

3. *The mapping h is an isomorphism from the quotient structure $(L(\mathcal{A}), (R'_i)_{1 \leq i \leq k})/E_\equiv$ onto $\mathcal{M}$.*

The ω-tree-automatic presentation is said to be injective if the equivalence relation $E_\equiv$ is just the equality relation on $L(\mathcal{A})$. In this case $\mathcal{A}_=$ and $E_\equiv$ can be omitted and h is simply an isomorphism from $(L(\mathcal{A}), (R'_i)_{1 \leq i \leq k})$ onto $\mathcal{M}$. A relational structure is said to be (injectively) ω-tree-automatic if it has an (injective) ω-tree-automatic presentation.

We now recall a recent decidability result of [BKR09] about injectively ω-tree-automatic structures.

The quantifier $\exists$ is usual in first-order logic. In addition we can consider the infinity quantifier $\exists^\infty$, and the cardinality quantifiers $\exists^\kappa$ for any cardinal κ. The set of first-order formulas is denoted by FO. The set of formulas using the additional quantifiers $\exists^\infty$, $\exists^{\aleph_0}$ and $\exists^{2^{\aleph_0}}$ is denoted by $\mathrm{FO}(\exists^\infty, \exists^{\aleph_0}, \exists^{2^{\aleph_0}})$. If $\mathcal{L}$ is a set of formulas, the $\mathcal{L}$-theory of a structure $\mathcal{M}$ is the set of sentences (i.e., formulas without free variables) in $\mathcal{L}$ that hold in $\mathcal{M}$.

If $\mathcal{M}$ is a relational structure in a signature τ which contains only relational symbols and equality, and ψ is a formula in $\mathrm{FO}(\exists^\infty, \exists^{\aleph_0}, \exists^{2^{\aleph_0}})$, then the semantics of the above quantifiers are defined as follows.

- $\mathcal{M} \models \exists^\infty x\ \psi$ if and only if there are infinitely many $a \in \mathcal{M}$ such that $\mathcal{M} \models \psi(a)$.
- $\mathcal{M} \models \exists^\kappa x\ \psi$ if and only if the set $\psi^{\mathcal{M}} = \{a \in \mathcal{M} \mid \mathcal{M} \models \psi(a)\}$ has cardinality κ.

Theorem 2.2 ([BKR09]) *Let $n \geq 1$ be an integer. If $\mathcal{M}$ is an injectively ω-tree-automatic structure then its $\mathrm{FO}(\exists^\infty, \exists^{\aleph_0}, \exists^{2^{\aleph_0}})$-theory is decidable. Moreover assume that $\mathcal{M} = (M, (R_i^M)_{1\leq i\leq k})$ is a relational structure and an injective ω-tree-automatic presentation of the structure $\mathcal{M}$ is formed by a tuple of Muller tree automata $(\mathcal{A}, (\mathcal{A}_i)_{1\leq i\leq k})$ with an isomorphism h from $L(\mathcal{A})$ onto M. If $\psi(x_1, \ldots, x_p)$ is a formula in $\mathrm{FO}(\exists^\infty, \exists^{\aleph_0}, \exists^{2^{\aleph_0}})$, with free variables $x_1, \ldots, x_p$, then the set $h^{-1}[\{(a_1, \ldots, a_p) \in M^p \mid \mathcal{M} \models \psi(a_1, \ldots, a_p)\}]$ is a regular subset of $L(\mathcal{A})^p$ and one can effectively construct a tree automaton accepting it.*

3 Injectively ω-tree-automatic ordinals

We now give a complete characterization of the class of injectively ω-tree-automatic ordinals.

Theorem 3.1 *An ordinal α is an injectively ω-tree-automatic ordinal if and only if it is smaller than the ordinal ω^{ω^ω}.*

Proof. As usual, an ordinal α is considered as a linear order $(\alpha, <)$. We first asssume that the ordinal $(\alpha, <)$ has an injective ω-tree-automatic presentation, which is given by two Muller tree automata $(\mathcal{A}, \mathcal{A}_<)$ and an isomorphism h from $L(\mathcal{A})$ onto α. We now distinguish two cases.

First Case. The ordinal α is countable. Then $L(\mathcal{A})$ is countable and then it is proved by Niwinski in [Niw91] that the infinite trees of the tree language $L(\mathcal{A})$ can be represented by *finite* binary trees and that the corresponding language of finite trees is regular (accepted by a tree automaton reading labelled finite trees). Thus the structure $(\alpha, <)$ is not only injectively ω-tree-automatic but also injectively tree-automatic, the relation $<$ being also accepted by a tree automaton reading *finite* binary trees. But Delhommé proved in [Del04] that the tree-automatic ordinals are the ordinals smaller than the ordinal ω^{ω^ω}. Therefore in that case the ordinal α is smaller that the ordinal ω^{ω^ω}.

On the other hand it is proved in [CL07] that every tree-automatic structure is actually *injectively* tree-automatic and hence also *injectively* ω-tree-automatic. Thus every ordinal smaller that ω^{ω^ω} is also injectively ω-tree-automatic.

Thus the *countable* injectively ω-tree-automatic ordinals are the ordinals smaller than ω^{ω^ω}.

Second Case. The ordinal α is uncountable. Recall that Niwinski proved in [Niw91] that a regular language of infinite binary trees is either countable or has the cardinal $2^{\aleph_0}$ of the continuum and that one can decide, from a tree automaton $\mathcal{A}$, whether the tree language $L(\mathcal{A})$ is countable. Since there is an isomorphism h from $L(\mathcal{A})$ onto the uncountable ordinal α, the tree language $L(\mathcal{A})$ and the ordinal α have cardinality $2^{\aleph_0}$. In particular, the ordinal α is greater than or equal to the first uncountable ordinal ω_1. But then the initial segment of length ω_1 of α is definable by the formula $\phi(x) = \exists^{\aleph_0} y(y < x) \vee \neg\exists^{\infty} y(y < x)$. We can now infer from Theorem 2.2 that one can effectively construct some Muller tree automata $\mathcal{B}$ and $\mathcal{B}_<$ such that $L(\mathcal{B}) \subseteq L(\mathcal{A})$ and $(\mathcal{B}, \mathcal{B}_<)$ form an injective ω-tree-automatic presentation of the ordinal $(\omega_1, <)$. Moreover the formula $\forall x\ \phi(x)$ is satisfied in the structure $(\omega_1, <)$ and this can be deduced effectively from the two tree automata $\mathcal{B}$ and $\mathcal{B}_<$ of the injective ω-tree-automatic presentation of the structure. The ordinal $(\omega_1, <)$ being uncountable and injectively ω-tree-automatic it follows as above that it has the cardinality of the continuum and thus that $2^{\aleph_0} = \aleph_1$, i.e. that the continuum hypothesis CH is satisfied.

We have seen that we can construct $(\mathcal{B}, \mathcal{B}_<)$, with an isomorphism h' from $L(\mathcal{B})$ onto ω_1, which form an injective ω-tree-automatic presentation of the ordinal $(\omega_1, <)$. Recall that the usual axiomatic system ZFC is Zermelo-Fraenkel system ZF plus the axiom of choice AC. A model $(\mathbf{V}, \in)$ of the axiomatic system ZFC is a collection $\mathbf{V}$ of sets, equipped with the membership relation $\in$, where "$x \in y$" means that the set x is an element of the set y, which satisfies the axioms of ZFC. We often say " the model $\mathbf{V}$" instead of "the model $(\mathbf{V}, \in)$". The axioms of ZFC express some natural facts that we consider to hold in the universe of sets, and the axiomatic system ZFC is a commonly accepted framework in which all usual mathematics can be developed. We are here supposed to live in a universe $\mathbf{V}$ of sets which is a model of ZFC in which the continuum hypothesis is satisfied. But we know that, using the method of forcing developed by Cohen in 1963 to prove the consistency of the negation of the continuum hypothesis, one can show that there exists a forcing extension $\mathbf{V[G]}$ of $\mathbf{V}$ which is a model of ZFC in which $2^{\aleph_0} > \aleph_1$, see [Jec02]. Consider now the pair of Muller tree automata $(\mathcal{B}, \mathcal{B}_<)$ in this new model $\mathbf{V[G]}$ of ZFC. By [Niw91] the tree language $L(\mathcal{B})$ is still uncountable (because this can be decided by an algorithm defined in ZFC and hence does not depend on the ambient model of ZFC) and so it has the cardinality $2^{\aleph_0}$ of the continuum. Moreover the automaton $\mathcal{B}_<$ defines a linear order $<_0$ on $L(\mathcal{B})$ such that every initial segment is countable (because this is expressed by the sentence $\forall x\ \phi(x)$ in $\mathrm{FO}(\exists^\infty, \exists^{\aleph_0}, \exists^{2^{\aleph_0}})$ which was satisfied in the structure $(\omega_1, <)$ in the universe $\mathbf{V}$) . But one can show that this is in contradiction with the negation $2^{\aleph_0} > \aleph_1$ of the continuum hypothesis. Indeed we can construct by transfinite induction a strictly increasing infinite sequence $(x_\alpha)_{\alpha<\omega_1}$ in $L(\mathcal{B})$ such that each initial segment $I_\alpha = \{x \in L(\mathcal{B}) \mid x <_0 x_\alpha\}$ is countable. Consider now the union $I = \bigcup_{\alpha<\omega_1} I_\alpha$. The set I forms an initial segment of $L(\mathcal{B})$ for the linear order $<_0$ and it has cardinality $\aleph_1 < 2^{\aleph_0}$. Then there exists an element $y \in L(\mathcal{B})$ such that $x_\alpha < y$ for each $\alpha < \aleph_1$. But then the initial segment $\{x \in L(\mathcal{B}) \mid x <_0 y\}$ is uncountable and this leads to a contradiction. Finally we have proved that there are no uncountable injectively ω-tree-automatic ordinals. $\square$

4 The hierarchy of ω^n-automatic structures

Let Σ be a finite alphabet, and α be an ordinal; a word of length α (or α-word) over the alphabet Σ is an α-sequence $(x(\beta))_{\beta<\alpha}$ (or sequence of length α) of letters in Σ. The set of α-words over the alphabet Σ is denoted by Σ^α. The concatenation of an α-word $x = (x(\beta))_{\beta<\alpha}$ and of a γ-word $y = (y(\beta))_{\beta<\gamma}$ is the $(\alpha+\gamma)$-word $z = (z(\beta))_{\beta<\alpha+\gamma}$ such that $z(\beta) = x(\beta)$ for $\beta < \alpha$ and $z(\beta) = y(\beta')$ for $\alpha \leq \beta = \alpha + \beta' < \alpha + \gamma$; it is denoted $z = x \cdot y$ or simply $z = xy$.

We recall now the definition and behaviour of automata reading words of ordinal length

Definition 4.1 ([Woj84, Woj85, Bed96]) *An ordinal Büchi automaton is a sextuple* $(\Sigma, Q, q_0, \Delta, \gamma, F)$*, where* Σ *is a finite alphabet,* Q *is a finite set* *of states, $q_0 \in Q$ is the initial state, $\Delta \subseteq Q \times \Sigma \times Q$ is the transition relation for successor steps, $\gamma \subseteq P(Q) \times Q$ is the transition relation for limit steps, and $F \subseteq Q$ is the set of accepting states.*

A run of the ordinal Büchi automaton $\mathcal{A} = (\Sigma, Q, q_0, \Delta, \gamma, F)$ reading a word σ of length α, is an $(\alpha + 1)$-sequence of states x defined by: $x(0) = q_0$ and, for $i < \alpha$, $(x(i), \sigma(i), x(i+1)) \in \Delta$ and, for i a limit ordinal, $(Inf(x,i), x(i)) \in \gamma$, where $Inf(x,i)$ is the set of states which cofinally appear during the reading of the i first letters of σ, i.e.

$$Inf(x,i) = \{q \in Q \mid \forall \mu < i, \exists \nu < i \text{ such that } \mu < \nu \text{ and } x(\nu) = q\}$$

A run x of the automaton $\mathcal{A}$ over the word σ of length α is called successful if $x(\alpha) \in F$. A word σ of length α is accepted by $\mathcal{A}$ if there exists a successful run of $\mathcal{A}$ over σ. We denote $L_\alpha(\mathcal{A})$ the set of words of length α which are accepted by $\mathcal{A}$. An α-language L is a regular α-language if there exists an ordinal Büchi automaton $\mathcal{A}$ such that $L = L_\alpha(\mathcal{A})$.

An ω^n-automaton is an ordinal Büchi automaton reading only words of length ω^n for some integer $n \geq 1$.

Recall that we can obtain regular ω^n-languages from regular ω-languages and regular ω^{n-1}-languages by the use of the notion of substitution.

Proposition 4.2 (see [Fin01]) *Let $n \geq 2$ be an integer. An ω^n-language $L \subseteq \Sigma^{\omega^n}$ is regular iff it is obtained from a regular ω-language $R \subseteq \Gamma^\omega$ by substituting in every ω-word $\sigma \in R$ each letter $a \in \Gamma$ by a regular ω^{n-1}-language $L_a \subseteq \Sigma^\omega$.*

We can obtain the following stronger result which will be useful in the sequel.

Proposition 4.3 *Let $n \geq 2$ be an integer. An ω^n-language $L \subseteq \Sigma^{\omega^n}$ is regular iff it is obtained from a regular ω-language $R \subseteq \Gamma^\omega$ by substituting in every ω-word $\sigma \in R$ each letter $a \in \Gamma$ by a regular ω^{n-1}-language $L_a \subseteq \Sigma^\omega$, where for all letters $a, b \in \Gamma$, $a \neq b$, the languages L_a and L_b are disjoint.*

Proof. The result follows easily from Proposition 4.2 and the two following facts: (1) The class of regular ω^n-languages is effectively closed under finite union, finite intersection, and complementation, i.e. we can effectively construct, from two ω^n-automata $\mathcal{A}$ and $\mathcal{B}$, some ω^n-automata $\mathcal{C}_1$, $\mathcal{C}_2$, and $\mathcal{C}_3$, such that $L(\mathcal{C}_1) = L(\mathcal{A}) \cup L(\mathcal{B})$, $L(\mathcal{C}_2) = L(\mathcal{A}) \cap L(\mathcal{B})$, and $L(\mathcal{C}_3)$ is the complement of $L(\mathcal{A})$. (2) The class of regular ω-languages $R \subseteq \Gamma^\omega$ is effectively closed under the substitutions $\Phi : \Gamma \to P(X)$, where $X \subseteq \Gamma$ is a finite alphabet and $P(X)$ is the powerset of X. $\square$

We have defined in [FT12] the notion of ω^n-automatic presentation of a structure and of ω^n-automatic structure which is simply obtained by replacing in the above Definition 2.1 the Muller tree automata by some ω^n-automata.

Recently Schlicht and Stephan have independently considered *finite word* α-automatic structures which are relational structures whose domain and relations are accepted by automata reading *finite* α-words: they define *finite* α-words as words x of length α over an alphabet Σ containing a special symbol $\diamondsuit$ such that all but finitely many letters of x are equal to $\diamondsuit$. In particular they have proved the following result.

Theorem 4.4 *An ordinal is finite word ω^n-automatic, where $n \geq 1$ is an integer, iff it is smaller than the ordinal ω^{ω^n}.*

We now extend this result to the class of injectively ω^n-automatic structures considered in [FT12].

Theorem 4.5 *An ordinal is injectively ω^n-automatic, where $n \geq 1$ is an integer, iff it is smaller than the ordinal ω^{ω^n}.*

Proof. Notice first that Schlicht and Stephan considered only injective presentations of structures. Thus Theorem 4.4 implies that every ordinal $\alpha < \omega^{\omega^n}$ is injectively ω^n-automatic.

On the other hand we proved in [FT12] that every injectively ω^n-automatic structure is also an injectively ω-tree-automatic structure. Thus it follows from the above Theorem 3.1 that every injectively ω^n-automatic ordinal is countable.

But we now prove the following result.

Proposition 4.6 *All injectively ω^n-automatic countable ordinals are finite word ω^n-automatic.*

Proof. It suffices to show that if $L \subseteq \Sigma^{\omega^n}$ is a countable regular ω^n-language then the ω^n-words in L can be represented by *finite* ω^n-words. This is clear for $n = 1$ because a countable regular ω-language $L \subseteq \Sigma^\omega$ is of the form $L = \cup_{1 \leq j \leq n} U_i \cdot v_i^\omega$, where for each integer $i \in [1, n]$ $U_i \subseteq \Sigma^\star$ is a finitary language and v_i is a non-empty finite word over Σ. Then the general result can be proved by an easy induction on n, using Proposition 4.3. $\square$

Finally Theorem 4.5 follows from Theorem 4.4 and Proposition 4.6. $\square$

Corollary 4.7 *The hierarchy of injectively ω^n-automatic structures, $n \geq 1$, which was considered in [FT12], is strict.*

Remark 4.8 *Schlicht and Stephan asked whether a countable α-automatic structure is finite word α-automatic. The proof of Proposition 4.6 about ordinals extends easily to the general case of (injectively) ω^n-automatic structures. Thus any countable (injectively) ω^n-automatic structure is finite word ω^n-automatic.*

5 Concluding remarks

We have determined the classes of injectively ω-tree-automatic ordinals and of injectively ω^n-automatic ordinals, where $n \geq 2$ is an ordinal. This way we have also proved that the hierarchy of injectively ω^n-automatic structures, $n \geq 1$, which was considered in [FT12], is strict.

The next step in this research would be to determine the classes of ω-tree-automatic and ω^n-automatic ordinals, in the non-injective case. Recall that every ω^n-automatic structure is a Borel structure, i.e. has a Borel presentation, see [FT12], and thus it follows from a result of Harrington and Shelah in [HS82] that there are no uncountable ω^n-automatic ordinals. We conjecture that a countable ω^n-automatic structure is actually injectively ω^n-automatic. From this it would follow that the ω^n-automatic ordinals are also the ordinals smaller than the ordinal ω^{ω^n}.

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